MATH 1A - MOCK FINAL DELUXE - SOLUTIONS

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1. *(10 points, 5 points each)* Find the following limits

(a)

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+x} - x &= \lim_{x\to\infty} \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x}}+x} \\
&= \lim_{x\to\infty} \frac{x}{x\sqrt{1+\frac{1}{x}}+x} \qquad \text{since } \sqrt{x^2} = |x| = x, \text{ since } x > 0 \\
&= \lim_{x\to\infty} \frac{x}{x\left(\sqrt{1+\frac{1}{x}}+1\right)} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} \\
&= \frac{1}{1+1} \\
&= \frac{1}{2}
\end{aligned}$$

(b)

$$\lim_{x\to\infty} \frac{(\ln(x))^2}{x} \overset{H}{=} \lim_{x\to\infty} \frac{2\ln(x)\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln(x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = \frac{2}{\infty} = 0$$

Date: Friday, August 5th, 2011.

2. *(10 points)* Use the **definition** of the derivative to calculate $f'(x)$, where:

$$f(x) = \frac{1}{x}$$

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} \\
&= \lim_{x \to a} \frac{\frac{a}{ax} - \frac{x}{ax}}{x - a} \\
&= \lim_{x \to a} \frac{\frac{a-x}{ax}}{x - a} \\
&= \lim_{x \to a} \frac{a - x}{ax(x - a)} \\
&= \lim_{x \to a} \frac{-(x - a)}{ax(x - a)} \\
&= \lim_{x \to a} \frac{-1}{ax} \\
&= -\frac{1}{a^2}
\end{aligned}$$

Hence $\boxed{f'(x) = -\frac{1}{x^2}}$

3. *(10 points, 5 points each)* Find the derivatives of the following functions

   (a) $f(x) = x^{\cos(x)}$
   Logarithmic differentiation

   1) Let $y = x^{\cos(x)}$

   2) Then $\ln(y) = \cos(x)\ln(x)$

   3) $\frac{y'}{y} = -\sin(x)\ln(x) + \frac{\cos(x)}{x}$

   4)

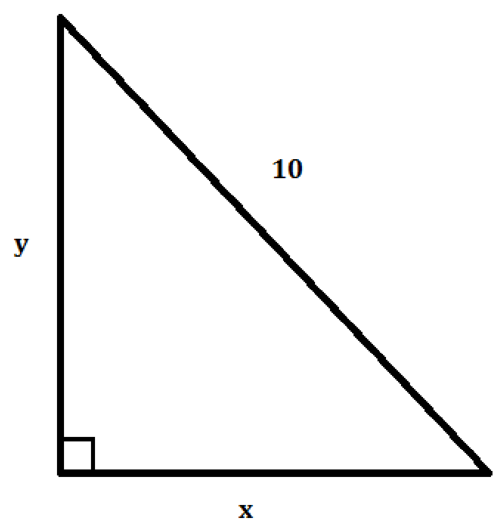
$$y' = y\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right) = x^{\cos(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)$$

   (b) y', where $x^3 + y^3 = xy$

$$\begin{aligned} 3x^2 + 3y^2y' &= y + xy' \\ 3y^2y' - xy' &= y - 3x^2 \\ (3y^2 - x)y' &= y - 3x^2 \\ y' &= \frac{y - 3x^2}{3y^2 - x} \end{aligned}$$

4. *(15 points)* A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder is sliding away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

   1) <u>Picture:</u>
   1A/Math 1A Summer/Exams/MockFDladder.png



   2) WTF $\frac{dy}{dt}$ when $x = 6$

   3) By the Pythagorean theorem: $x^2 + y^2 = 10^2$.

   4) Differentiating, we get: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

   5) However, $x = 6$, $\frac{dx}{dt} = 1$. Moreover, if you re-draw the same triangle with $x = 6$ plugged in, you should notice that it's an $6 - 8 - 10$ triangle, so $y = 8$, and hence:

$$
\begin{aligned}
2(6)(1) + 2(8)\frac{dy}{dt} &= 0 \\
12 + 16\frac{dy}{dt} &= 0 \\
\frac{dy}{dt} &= -\frac{16}{12} \\
\frac{dy}{dt} &= -\frac{4}{3}
\end{aligned}
$$

6) So $\boxed{\frac{dy}{dt} = -\frac{4}{3} \text{ ft/s}}$

5. *(20 points)* If $12\pi\ cm^2$ of material is available to make a cylinder with an open top, find the largest possible volume of the cylinder.

   1) Picture: Just draw a picture of a cylinder with an open top. The base radius is r and the height is h.

   2) $V = \pi r^2 h$

      However, we know $S = 12\pi$, but $S = \pi r^2 + 2\pi r h$, so:

$$\begin{aligned} \pi r^2 + 2\pi r h =& 12\pi \\ r^2 + 2rh =& 12 \\ 2rh =& 12 - r^2 \\ h =& \frac{12 - r^2}{2r} \\ h =& \frac{6}{r} - \frac{r}{2} \end{aligned}$$

      Hence, $V(r) = \pi r^2 \left(\frac{6}{r} - \frac{r}{2}\right) = 6\pi r - \frac{\pi}{2} r^3$.

      So $\boxed{V(r) = 6\pi r - \frac{\pi}{2} r^3}$

   3) Constraint: $r > 0$.

   4) $V'(r) = 6\pi - \frac{3\pi}{2} r^2$

$$\begin{aligned} V'(r) =& 0 \\ 6\pi - \frac{3\pi}{2} r^2 =& 0 \\ 6\pi =& \frac{3\pi}{2} r^2 \\ 6 =& \frac{3}{2} r^2 \\ r^2 =& (6)\frac{2}{3} \\ r^2 =& 4 \\ r =& 2 \end{aligned}$$

By FDTAEV, $\boxed{r = 2}$ is the maximizer of V, and the largest possible volume is:

$$V(2) = 6\pi(2) - \frac{\pi}{2}(2)^3 = 12\pi - 4\pi = 8\pi$$

6. *(15 points)* Show that the following equation has exactly one solution in [-1,1]

$$x^4 - 5x + 1 = 0$$

Let $f(x) = x^4 - 5x + 1$

At least one solution: $f(0) = 1 > 0$, $f(1) = 1 - 5 + 1 = -3 < 0$, f is continuous, so by **the IVT**, f has at least one zero in $[-1, 1]$

At most one solution: Suppose f has at least two zeros a and b. Then $f(a) = f(b) = 0$, so by **Rolle's theorem**, there is some c in $(-1, 1)$ with $f'(c) = 0$.

But $0 = f'(c) = 4c^3 - 5$, so $4c^3 - 5 = 0$, so $c^3 = \frac{5}{4}$, so $c = \sqrt[3]{\frac{5}{4}} > 1$, which contradict the fact that c is in $(-1, 1)$, $\Rightarrow\Leftarrow$. Hence f has at most one zero in $[-1, 1]$.

Therefore, f has exactly one zero in $[-1, 1]$.

7. *(20 points)* Use the **definition** of the integral to find:

$$\int_1^2 x^2 dx$$

You may use the following formulas:

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Preliminary work:

- $f(x) = x^2$
- $a = 1$, $b = 2$, $\Delta x = \frac{2-1}{n} = \frac{1}{n}$
- $x_i = 1 + \frac{i}{n}$

$$\begin{aligned}
\int_1^2 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^n \Delta x f(x_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left(\frac{1}{n}\right)\left(1+\frac{i}{n}\right)^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left(\frac{1}{n}\right)\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}+\frac{2i}{n^2}+\frac{i^2}{n^3} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}+\sum_{i=1}^n \frac{2i}{n^2}+\sum_{i=1}^n \frac{i^2}{n^3} \\
&= \lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^n 1\right)+\frac{2}{n^2}\left(\sum_{i=1}^n i\right)+\frac{1}{n^3}\left(\sum_{i=1}^n i^2\right) \\
&= \lim_{n\to\infty} \frac{1}{n}(n)+\frac{2}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to\infty} 1+\frac{n+1}{n}+\frac{(n+1)(2n+1)}{6n^2} \\
&=1+1+\frac{2}{6} \\
&=2+\frac{1}{3} \\
&=\frac{7}{3}
\end{aligned}$$

Check: (not required, but useful)

$$\int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3}-\frac{1}{3}=\frac{7}{3}$$

8. *(30 points, 5 points each)* Find the following:

(a) The antiderivative F of $f(x) = x^2 + 3x^3 - 4x^7$ which satisfies $F(0) = 1$
The MGAD of f is:

$$F(x) = \frac{x^3}{3} + \frac{3x^4}{4} - \frac{4x^8}{8} + C = \frac{x^3}{3} + \frac{3}{4}x^4 - \frac{1}{2}x^8 + C$$

To solve for C, use the fact that $F(0) = 1$, so $0+0-0+C = 1$, so $\boxed{C = 1}$, and hence:

$$F(x) = \frac{x^3}{3} + \frac{3}{4}x^4 - \frac{1}{2}x^8 + 1$$

(b) $\int_{-1}^{1} |x|\, dx$ (**Hint:** Draw a picture)
If you draw a picture of $f(x) = |x|$, you should notice that the integral is the sum of two triangles, one with base 1 and height 1 (from -1 to 0) and the other one with base 1 and height 1 (from 0 to 1), hence we get:

$$\int_{-1}^{1} |x|\, dx = \frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) = \frac{1}{2} + \frac{1}{2} = 1$$

(c)

$$\int x^2 + 1 + \frac{1}{x^2+1} dx = \frac{x^3}{3} + x + \tan^{-1}(x) + C$$

(d) $\int_1^e \frac{(\ln(x))^2}{x} dx$
Let $u = \ln(x)$, then $du = \frac{1}{x}dx$, and $u(1) = \ln(1) = 0$, and $u(e) = \ln(e) = 1$, so:

$$\int_1^e \frac{(\ln(x))^2}{x} dx = \int_0^1 u^2 du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

(e) $g'(x)$, where $g(x) = \int_x^{e^x} \sqrt{1+t^2}dt$
Let $f(t) = \sqrt{1+t^2}$, then $g(x) = F(e^x) - F(x)$, so:

$$g'(x) = F'(e^x)e^x - F'(x) = f(e^x)e^x - f(x) = \sqrt{1+(e^x)^2}(e^x) - \sqrt{1+x^2}$$

(f) The average value of $f(x) = \sin(x)$ on $[-\pi, \pi]$

$$\frac{\int_{-\pi}^{\pi} \sin(x)dx}{\pi - (-\pi)} = \frac{0}{2\pi} = 0$$

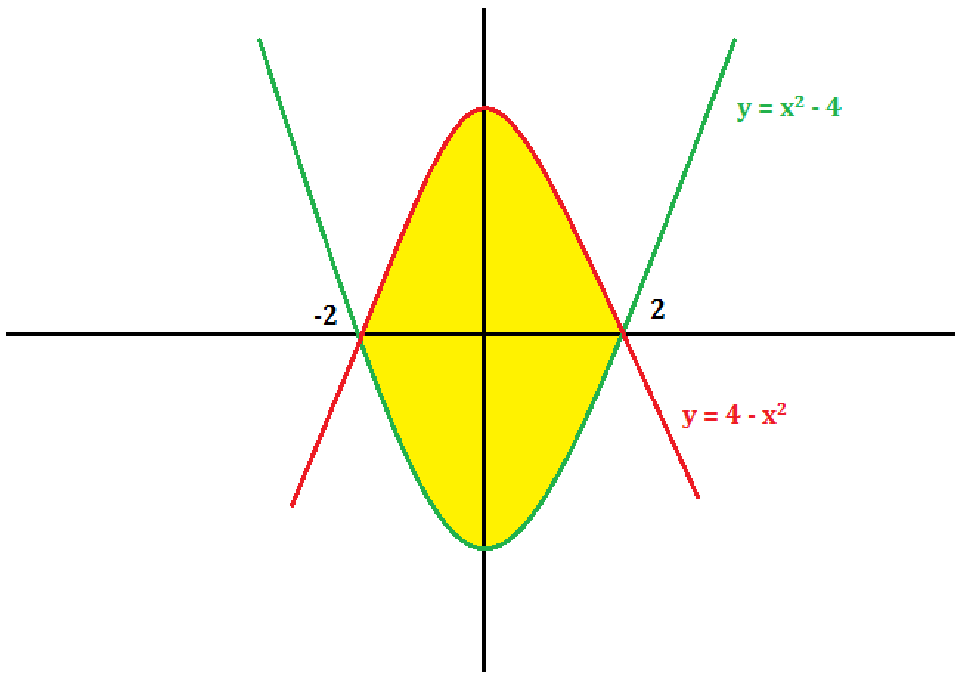
Since $\sin(x)$ is an odd function!

9. *(10 points)* Find the area of the region enclosed by the curves:

$$y = x^2 - 4 \quad \text{and} \quad y = 4 - x^2$$

First draw a picture:

1A/Math 1A Summer/Exams/MockFparabola.png



Then determine the points of intersection between the two parabolas:

$$\begin{aligned} x^2 - 4 =& 4 - x^2 \\ 2x^2 =& 8 \\ x^2 =& 4 \\ x =& \pm 2 \end{aligned}$$

And notice that on $[-2, 2]$, $4 - x^2$ is always above $x^2 - 4$, so the area of the region is:

$$\begin{aligned}\int_{-2}^{2}(4-x^2)-(x^2-4)dx =& \int_{-2}^{2} 8-2x^2dx \\ =& \left[8x-\frac{2}{3}x^3\right]_{-2}^{2} \\ =& 16-\frac{2}{3}(8)-(-16+\frac{2}{3}(8)) \\ =& 16-\frac{16}{3}+16-\frac{16}{3} \\ =& 32-\frac{32}{3} \\ =& \frac{64}{3}\end{aligned}$$

Of course, if you're clever about this, you might have noticed that the area is $4\int_0^2 4-x^2dx$, but you didn't have to be so clever about it! :)

10. *(10 points)* If $f(x) = x\ln(x)$, find:

(a) Intervals of increase and decrease, and local max/min

$f'(x) = \ln(x) + \frac{x}{x} = \ln(x) + 1$.

First of all:

$$f'(x) = 0 \Longleftrightarrow \ln(x) + 1 = 0 \Longleftrightarrow \ln(x) = -1 \Longleftrightarrow x = e^{-1} = \frac{1}{e}$$

Then, drawing a sign table if necessary, we see that:

$\boxed{f \text{ is decreasing on } \left(0, \frac{1}{e}\right) \text{ and increasing on } \left(\frac{1}{e}, \infty\right)}$ (careful about the domain of f!)

In particular, f has a local minimum at $\frac{1}{e}$, and $f\left(\frac{1}{e}\right) = \frac{1}{e}\ln\left(\frac{1}{e}\right) = -\frac{1}{e}$

(b) Intervals of concavity and inflection points

$f''(x) = \frac{1}{x}$.

In particular, since the domain of f is $(0, \infty)$, $x > 0$, so $f''(x) > 0$, hence $\boxed{f \text{ is concave up on } (0, \infty)}$. There are no inflection points.

Bonus 1 *(5 points)* Show that if f is continuous on $[0,1]$, then $\int_0^1 f(x)dx$ is bounded, that is, there are numbers m and M such that:

$$m \leq \int_0^1 f(x)dx \leq M$$

Hint: Use one of the 'value' theorems that we haven't used a lot in this course (see section 4.1)

By the extreme value theorem, f attains an absolute max M and an absolute min m. This means that for all x in $[0,1]$:

$$m \leq f(x) \leq M$$

Now integrate:

$$\int_0^1 m dx \leq \int_0^1 f(x)dx \leq \int_0^1 M dx$$

$$m(1-0) \leq \int_0^1 f(x)dx \leq M(1-0)$$

$$m \leq \int_0^1 f(x)dx \leq M$$

Bonus 2 *(5 points)* If $f(x) = Ax^3 + Bx^2 + Cx + D$ is a polynomial such that:

$$\frac{A}{4} + \frac{B}{3} + \frac{C}{2} + D = 0$$

Show that f has at least one zero on $(0, 1)$.

Hint: What is the *average* value of f on $[0, 1]$?

By the MVT for integrals on $[0, 1]$, for some c in (0,1), we have:

$$f(c) = \frac{\int_0^1 f(x)dx}{1 - 0}$$

But:

$$\begin{aligned}
\frac{\int_0^1 f(x)dx}{1-0} &= \int_0^1 f(x)dx \\
&= \int_0^1 (Ax^3 + Bx^2 + Cx + D)dx \\
&= \left[\frac{A}{4}x^4 + \frac{B}{3}x^3 + \frac{C}{2}x^2 + Dx\right]_0^1 \\
&= \frac{A}{4} + \frac{B}{3} + \frac{C}{2} + D \\
&= 0
\end{aligned}$$

Hence, for some c in (0,1), we have $\boxed{f(c) = 0}$, so f has at least one zero c in $(0, 1)$.

Bonus 3 *(5 points)* Another way to define $\ln(x)$ is:

$$\ln(x) = \int_1^x \frac{1}{t} dt$$

Show **using this definition only** that for all a and b:

$$\ln(ab) = \ln(a) + \ln(b)$$

Hint: Fix a constant a, and consider the function:

$$g(x) = \ln(ax) - \ln(x) - \ln(a)$$

$$\begin{aligned}
g(x) =& \ln(ax) - \ln(x) - \ln(a) \\
=& \int_1^{ax} \frac{1}{t} dt - \int_1^x \frac{1}{t} dt - \int_1^a \frac{1}{t} dt \\
=& F(ax) - F(1) - (F(x) - F(1)) - (F(a) - F(1))
\end{aligned}$$

Where F is an antiderivative of $f(t) = \frac{1}{t}$

Now differentiating g, and using the fact that a is a constant, we get:

$$\begin{aligned}
g'(x) =& F'(ax)(a) - 0 - F'(x) + 0 - 0 + 0 \\
=& f(ax)(a) - f(x) \\
=& \left(\frac{1}{ax}\right)(a) - \frac{1}{x} \\
=& \frac{1}{x} - \frac{1}{x} \\
=& 0
\end{aligned}$$

Hence $g'(x) = 0$, so $g(x) = C$, where C is a constant.

To figure out what C is, let's calculate $g(1)$:

$$
\begin{aligned}
g(1) =& C \\
\int_1^1 \frac{1}{t} dt =& C \\
0 =& C \\
C =& 0
\end{aligned}
$$

Hence $C = 0$, and so $g(x) = 0$, whence $\ln(ax) - \ln(x) - \ln(a) = 0$, so $\ln(ax) = \ln(a) + \ln(x)$.

Since this holds for all x, let $x = b$, and we get:

$$\ln(ab) = \ln(a) + \ln(b)$$

BAZINGA!!!